MATH1010E University Mathematics
Quiz 2
Suggested Solutions

1. (a)

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x^2}{x^2\sin x^2} &= \lim_{x\to 0}\frac{2x\sin x^2}{2x\sin x^2+2x^3\cos x^2}\\
&= \lim_{x\to 0}\frac{\sin x^2}{\sin x^2+x^2\cos x^2}\\
&= \lim_{x\to 0}\frac{2x\cos x^2}{4x\cos x^2-2x^3\sin x^2}\\
&= \lim_{x\to 0}\frac{\cos x^2}{2\cos x^2-x^2\sin x^2}\\
&= \frac{1}{2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 1} x^{\frac{1}{1-x}} &= \lim_{x\to 1} e^{\frac{\ln x}{1-x}}\\
&= e^{\lim_{x\to 1}\frac{\ln x}{1-x}}\\
&= e^{\lim_{x\to 1}\frac{1/x}{-1}}\\
&= e^{-1}.
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) &= \lim_{x\to 1}\frac{(x-1)-\ln x}{(x-1)\ln x}\\
&= \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}\\
&= \lim_{x\to 1}\frac{x-1}{x\ln x+(x-1)}\\
&= \lim_{x\to 1}\frac{1}{\ln x+1+1}\\
&= \frac{1}{2}.
\end{aligned}$$

(d)

$$\begin{aligned}
\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x} &= \lim_{x\to 0}\left(\frac{x}{\sin x}\cdot x\sin\frac{1}{x}\right)\\
&= \left(\lim_{x\to 0}\frac{x}{\sin x}\right)\cdot\left(\lim_{x\to 0} x\sin\frac{1}{x}\right)\\
&= 1\cdot 0 = 0.
\end{aligned}$$

2. (a) Differentiating gives
$$f'(x) = e^{-x}(1 - x^2).$$
Set $f'(x) = 0$, we obtain the critical point $x = \pm 1$. Computing the second derivative,
$$f''(x) = e^{-x}(x^2 - 2x - 1).$$
Using the second derivative test, $x = 1$ is a local maximum since $f''(1) = e^{-1}(-2) < 0$, and $x = -1$ is a local minimum since $f''(-1) = 2e > 0$.

   (b) Note that $f(0) = 1$ and
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (x + 1)^2 e^{-x} = 0,$$
and we have only one critical point $x = 1$ in the interval $[0, +\infty)$, with $f(1) = 4e^{-1} > 1$. Therefore, the minimum does not exist and the maximum is $4e^{-1}$ located at $x = 1$.

3. Without loss of generality, we can assume $x > y$. By mean value theorem, there exists $\xi \in (y, x)$ such that
$$\sin x - \sin y = (\cos\xi)(x - y).$$
Since $|\cos\xi| \le 1$, we conclude that $|\sin x - \sin y| \le |x - y|$.

4. Implicitly differentiating the equation gives
$$2x + 2y + 2xy' - 2yy' = 2,$$
which we can solve for y' to get
$$y' = \frac{1 - x - y}{x - y}. \tag{1}$$
At $(2, 0)$, we have
$$y' = \frac{1 - 2 - 0}{2 - 0} = -\frac{1}{2}.$$
Differentiating (1), we obtain
$$y'' = \frac{(x - y)(-1 - y') - (1 - x - y)(1 - y')}{(x - y)^2}.$$
Evaluate at $(0, 2)$, we get
$$y'' = \frac{2(-1 + \frac{1}{2}) - (1 - 2)(1 + \frac{1}{2})}{4} = \frac{1}{8}.$$

— End of Solutions to Quiz 2 —